The Language of Sequences

Things you should already know

Fact (Famous Sequences) —

Square numbers	$1,\ 4,\ 9,\ 16,\ 25,\ \ldots$
Cube numbers	$1,\ 8,\ 27,\ 64,\ 125,\ \ldots$
Triangular numbers	$1,\ 3,\ 6,\ 10,\ 15,\ \ldots$
Powers of 2	$1,\ 2,\ 4,\ 8,\ 16,\ \ldots$
Primes	$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ \ldots$
Fibonacci	$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots$

Notation

Definition. A sequence is an ordered list of numbers. Each number in the list is called a term.

We write U_n for the nth term of a sequence, so U_1 is the first term, U_2 the second, and so on. The position n is always a positive integer.

> **Remark.** $T(n)$, t_n, x_n and a_n are also in common use.

There are two fundamentally different ways to describe a sequence with a formula.

Definition (Position-to-term)**.** A position-to-term formula (or *direct formula*) gives U_n directly in terms of the position n.

$$\text{e.g.} \quad U_n = n^2 + 2 \quad \text{gives} \quad 3,\ 6,\ 11,\ 18,\ 27,\ \ldots$$

Definition (Term-to-term)**.** A term-to-term formula (or *iterative* / *inductive* definition) tells you how to get from each term to the next:

$$U_{n+1} = U_n + 4, \qquad U_1 = 3$$

An iterative formula is incomplete without a **starting value**.

Example

Write down the first five terms of each sequence.

1. $U_n = (-1)^n n^2$
2. $U_{n+1} = 2U_n + 1, \quad U_1 = 3$
3. $U_{n+1} = U_n + U_{n-1}, \quad U_1 = 3,\ U_2 = 4$

1. $-1,\ 4,\ -9,\ 16,\ -25$
2. $3,\ 7,\ 15,\ 31,\ 63$
3. $3,\ 4,\ 7,\ 11,\ 18$ *(two starting values are needed)*

Example

Find an iterative definition for each of these sequences.

1. 25, 23, 21, 19, 17, ...
2. 80, 40, 20, 10, 5, ...
3. 8, −12, 18, −27, 40.5, ...

1. $U_{n+1} = U_n - 2, \quad U_1 = 25$
2. $U_{n+1} = \frac{1}{2}U_n, \quad U_1 = 80$
3. $U_{n+1} = -1.5\,U_n, \quad U_1 = 8$

Example

A sequence is defined by $a_1 = 2, \quad a_{n+1} = \dfrac{1}{1 - a_n}$. Find a_{2026}.

$a_2 = \dfrac{1}{1-2} = -1, \quad a_3 = \dfrac{1}{1-(-1)} = \frac{1}{2}, \quad a_4 = \dfrac{1}{1-\frac{1}{2}} = 2$

The sequence is periodic with period 3. Since $2026 = 3 \times 675 + 1$, $a_{2026} = a_1 = 2$.

Example (Edexcel C1, adapted)

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 4, \qquad a_{n+1} = \frac{a_n}{a_n + 1}, \quad n \geq 1.$$

1. Find a_2, a_3 and a_4, writing your answers as simplified fractions.
2. Given that $a_n = \dfrac{4}{pn+q}$ for constants p and q, state the values of p and q.
3. Hence find the value of N such that $a_N = \dfrac{4}{321}$.

1. $a_2 = \dfrac{4}{5}, \quad a_3 = \dfrac{4/5}{9/5} = \dfrac{4}{9}, \quad a_4 = \dfrac{4/9}{13/9} = \dfrac{4}{13}$

2. The denominators $1, 5, 9, 13$ *go up by* 4 *each time:* $p = 4$, $q = -3$.

3. $4N - 3 = 321 \implies N = 81$.

Textbook Exercises: SPS Course 1.4, Exercise 1

Arithmetic Sequences

Definition. An arithmetic sequence is one in which the difference between consecutive terms is constant, e.g.

$$-3, \quad 2, \quad 7, \quad 12, \quad 17, \quad \ldots$$

We use the standard notation:

First term	a
Common difference	d
nth term	U_n

Here $a = -3$ and $d = 5$.

Theorem (The nth Term)
The nth term of an arithmetic sequence is

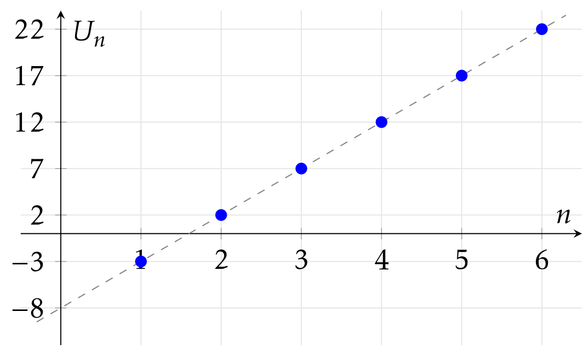
$$U_n = a + (n-1)d$$

Why $(n-1)$ and not n?

To get from the 1st term to the nth term you take $n-1$ steps of size d. For instance $U_5 = a + 4d$: four steps from a.

For $-3, 2, 7, 12, 17, \ldots$: $U_n = -3 + 5(n-1) = 5n - 8$.

Remark. Compare $U_n = 5n - 8$ with the line $y = 5x - 8$: the common difference is the gradient, and the "zeroth term" $a - d$ is the intercept. An arithmetic sequence is a straight line sampled at $n = 1, 2, 3, \ldots$



Example

For the sequence 2, 9, 16, 23, ...

1. Find the 50th term.
2. Which term of the sequence equals 625?

$a = 2$, $d = 7$, so $U_n = 7n - 5$.
1. $U_{50} = 7 \times 50 - 5 = 345$.
2. $7n - 5 = 625 \implies n = 90$: the 90th term.

Example

Which is the first term of the sequence -1, 3, 7, 11, ... to exceed 500?
Trial and improvement is not an acceptable method.

$U_n = 4n - 5$. We need $4n - 5 > 500$, i.e. $n > 126.25$.
The smallest integer n is 127, so the answer is the 127th term ($U_{127} = 503$).

Example (Unknowns in the terms)

The first three terms of an arithmetic sequence are

$$2x, \qquad x+4, \qquad 2x-7.$$

Find the value of x.

Equal differences: $(x+4)-2x = (2x-7)-(x+4)$

$4-x = x-11 \implies x = 7.5.$

Textbook Exercises: SPS Course 2.11, Exercises 1A and 1B

Arithmetic Series

Definition. When the terms of a sequence are *added*, we call it a series:

$$\text{sequence:} \quad -3,\ 2,\ 7,\ 12,\ \ldots \qquad \text{series:} \quad -3+2+7+12+\cdots$$

The phrase arithmetic progression (AP) is used for either. We write S_n for the sum of the first n terms.

Gauss's Trick

The young Gauss, asked to add the whole numbers from 1 to 100, produced the answer within seconds.

Example

Find $1+2+3+\cdots+100$ without adding up 100 numbers.

Write the sum forwards and backwards:

$$S = 1+2+3+\cdots+99+100$$
$$S = 100+99+98+\cdots+2+1$$

Adding in columns: $2S = 101\times 100$, *so* $S = 5050$.

The same pairing argument works in general.

$$S_n = a+[a+d]+[a+2d]+\cdots+[a+(n-1)d]$$
$$S_n = [a+(n-1)d]+[a+(n-2)d]+\cdots+[a+d]+a$$

Each column sums to $2a+(n-1)d$*, and there are* n *columns:*

$$2S_n = n\Big[2a+(n-1)d\Big] \implies S_n = \frac{n}{2}\Big[2a+(n-1)d\Big]$$

Theorem (Sum of an Arithmetic Series)

$$S_n = \frac{n}{2}\Big[2a + (n-1)d\Big]$$

Writing l for the last term, $l = a + (n-1)d$, this can also be written

$$S_n = \frac{n}{2}(a+l)$$

Remark. For integer a and d the formula always gives a whole number, despite the $\frac{n}{2}$. Why?

Example

Evaluate:

1. $101 + 102 + 103 + \cdots + 200$
2. $1 + 3 + 5 + \cdots + 99$
3. the sum of all multiples of 7 between 100 and 1000.

1. 100 terms, first 101, last 200: $S = \frac{100}{2}(101+200) = 15\,050$.
2. $n = 50$: $S = \frac{50}{2}(1+99) = 2500 = 50^2$. *The sum of the first n odd numbers is n^2.*
3. First $105 = 7 \times 15$, *last* $994 = 7 \times 142$, *so* $n = 128$: $S = \frac{128}{2}(105+994) = 70\,336$.

Example

Show that the sum of the first n positive even numbers is $n(n+1)$.

$a = 2$, $d = 2$: $S_n = \frac{n}{2}\Big[4 + 2(n-1)\Big] = \frac{n}{2}(2n+2) = n(n+1)$.

Example

How many terms of the arithmetic series $45+55+65+75+\cdots$ are needed for the sum to reach 3300?

$\frac{n}{2}\left[90+10(n-1)\right]=3300 \implies n(n+8)=660$

$n^2+8n-660=0 \implies (n+30)(n-22)=0 \implies n=22$ *(rejecting* $n=-30$*).*

Textbook Exercises: SPS Course 2.11, Exercises 2 and 3

Working Backwards

Exam questions rarely hand you a and d; recover them first.

Example

An arithmetic sequence has $U_{10} = 65$ and $U_{20} = 135$. Find U_{30}.

Method 1: $a + 9d = 65,\ \ a + 19d = 135$. *Subtracting:* $d = 7$, $a = 2$, *so* $U_{30} = 2 + 29 \times 7 = 205$.
Method 2: U_{10} *to* U_{20} *is 10 steps, worth* 70; *ten more:* $U_{30} = 135 + 70 = 205$.

Example

An arithmetic series has $S_{10} = -70$ and $S_{20} = -540$. Find the first term and the common difference.

$S_{10} = 5(2a + 9d) = -70 \implies 2a + 9d = -14$
$S_{20} = 10(2a + 19d) = -540 \implies 2a + 19d = -54$
Subtracting: $10d = -40$, *so* $d = -4$ *and* $a = 11$.

Example

The 8th term of an arithmetic progression is twice the 3rd term, and the sum of the first two terms is 350. Find a and d.

$a + 7d = 2(a + 2d) \implies a = 3d$
$S_2 = 2a + d = 350 \implies 7d = 350 \implies d = 50,\ a = 150.$

Sigma Notation

Definition. The symbol $\sum$ (capital sigma, for "Sum") compresses a series:

$$\sum_{r=1}^{n} U_r = U_1 + U_2 + U_3 + \cdots + U_n \qquad \text{e.g.} \quad \sum_{r=1}^{20} (3r+2) = 5 + 8 + 11 + \cdots + 62$$

Example (Edexcel C1)

Given that for all positive integers n,

$$\sum_{r=1}^{n} a_r = 12 + 4n^2,$$

1. find the value of $\sum_{r=1}^{5} a_r$;
2. find the value of a_6.

1. $S_5 = 12 + 4 \times 25 = 112.$
2. $a_6 = S_6 - S_5 = (12 + 144) - 112 = 44.$ *In general* $U_n = S_n - S_{n-1}$.

Example (Edexcel C1)

A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P. Salary increases by £$(2T)$ each year.
Scheme 2: Salary in Year 1 is £$(P+1800)$. Salary increases by £T each year.

1. Show that the total earned under Scheme 1 over the 10 years is £$(10P+90T)$.
2. For the 10-year period, the total earned is the same under both schemes. Find the value of T.
3. For this value of T, the salary in Year 10 under Scheme 2 is £29 850. Find the value of P.

1. $S_{10} = \frac{10}{2}\left[2P + 9(2T)\right] = 10P + 90T$.
2. Scheme 2: $S_{10} = \frac{10}{2}\left[2(P+1800) + 9T\right] = 10P + 18\,000 + 45T$.
Equal totals: $90T = 18\,000 + 45T \implies T = 400$.
3. Year 10 of Scheme 2: $(P+1800) + 9 \times 400 = 29\,850 \implies P = 24\,450$.

Textbook Exercises: SPS Course 2.11, Exercises 4 and 5

Quadratic and Cubic Sequences

Not every sequence has constant differences. For

$$3, \quad 6, \quad 11, \quad 18, \quad 27, \quad \ldots$$

the differences are $3, 5, 7, 9$ — not constant, but the *differences of the differences* are.

Definition. In a difference table we list the first differences between consecutive terms, then the second differences between those, and so on.

U_n	3		6		11		18		27
1st		3		5		7		9	
2nd			2		2		2		

Fact — • Constant **first** differences $\iff$ linear sequence $U_n = dn + c$, where d is the difference.

- Constant **second** differences $\iff$ quadratic sequence $U_n = an^2 + bn + c$, where

$$a = \frac{\text{second difference}}{2}.$$

- Constant **third** differences $\iff$ cubic sequence, with leading coefficient $\dfrac{\text{third difference}}{6}$.

Where do the 2 and the 6 come from? Build the difference table of $U_n = an^2 + bn + c$ in general:

$U_{n+1} - U_n = a(n+1)^2 + b(n+1) - an^2 - bn = a(2n+1) + b$, which is linear in n with gradient $2a$, so the second differences are constant at $2a$. The same computation for an^3 gives third differences of $6a$.

Finding the formula

Tip (Method for quadratic sequences) 1. Find the second difference; halve it to get a.

2. Subtract an^2 from each term of the sequence.
3. What remains is linear — find it by inspection.

Example

Find a formula for the nth term of:

1. 2, 11, 26, 47, 74, ...
2. 2.5, 4, 6.5, 10, 14.5, ...

1. First differences $9, 15, 21, 27$*; second differences constant at 6, so* $a = 3$.
Subtract $3n^2$ *(i.e.* $3, 12, 27, 48, 75$*):* $-1, -1, -1, -1, -1$. $U_n = 3n^2 - 1$.
2. Second differences constant at 1, so $a = \frac{1}{2}$.
Subtract $\frac{1}{2}n^2$ *(i.e.* $0.5, 2, 4.5, 8, 12.5$*):* $2, 2, 2, 2, 2$. $U_n = \frac{1}{2}n^2 + 2$.

Example

Find a formula for the nth term of 3, 11, 31, 69, 131, ...

First differences $8, 20, 38, 62$*; second differences* $12, 18, 24$*; third differences constant at 6, so the sequence is cubic with leading coefficient* $6/6 = 1$.
Subtract n^3 *(i.e.* $1, 8, 27, 64, 125$*):* $2, 3, 4, 5, 6$ *— which is* $n + 1$.
So $U_n = n^3 + n + 1$.

Example (Patterns can lie)

Place n points on a circle and join every pair with a chord. The maximum number of regions formed is:

n	1	2	3	4	5	6
regions	1	2	4	8	16	?

What is the next term?

*Not 32 — it is **31**. The formula is the quartic* $U_n = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$*, which a difference table will recover.*

Textbook Exercises: SPS Course 1.4, Exercise 2 and Revision Exercise 1.4 Q4

Problem Solving with Sequences

Example (Edexcel C1)

On John's 10th birthday he received the first of an annual birthday gift of money from his uncle. The first gift was £60, and on each subsequent birthday the gift was £15 more than the year before.

1. Show that, immediately after his 12th birthday, the total of these gifts was £225.
2. Find the amount John received on his 18th birthday.
3. Find the total received up to and including his 21st birthday.
4. When John had received n gifts, the total was £3375. Show that $n^2 + 7n = 25 \times 18$.
5. Hence determine John's age at this time.

$a = 60$, $d = 15$.
1. $S_3 = \frac{3}{2}(120 + 30) = 225$.
2. His 18th birthday is the 9th gift: $U_9 = 60 + 8 \times 15 = £180$.
3. 12 gifts: $S_{12} = \frac{12}{2}(120 + 11 \times 15) = 6 \times 285 = £1710$.
4. $\frac{n}{2}\left[120 + 15(n-1)\right] = 3375 \implies n(105 + 15n) = 6750 \implies n^2 + 7n = 450 = 25 \times 18$.
5. $(n+25)(n-18) = 0 \implies n = 18$: the 18th gift, so John is $9 + 18 = 27$.

Example (Edexcel C1)

A company making 140 bicycles each week plans to increase production by d each week — 140 in week 1, $140 + d$ in week 2, and so on — until it produces 206 in week 12.

1. Find the value of d.
2. After week 12 the company makes 206 bicycles each week. Find the total number made in the first 52 weeks.

1. $140 + 11d = 206 \implies d = 6$.

2. *Weeks 1–12:* $S_{12} = \frac{12}{2}(140 + 206) = 2076$. *Weeks 13–52:* $40 \times 206 = 8240$.
Total: $2076 + 8240 = 10\,316$.

Example (Edexcel C1)

Each year, Andy pays into a savings scheme: £600 in year one, increasing by £120 each year. Kim pays into a different scheme at the same time: £130 in year one, increasing by £80 each year.
At the end of year N, Andy has paid in, in total, exactly twice as much as Kim. Find the value of N.

Andy: $S_N = \frac{N}{2}\left[1200 + 120(N-1)\right] = \frac{N}{2}(1080 + 120N)$.
Kim: $S_N = \frac{N}{2}\left[260 + 80(N-1)\right] = \frac{N}{2}(180 + 80N)$.
Twice as much: $1080 + 120N = 2(180 + 80N) = 360 + 160N$
$720 = 40N \implies N = 18$.

Challenge Problems

Exercise. A staircase has n steps. You climb it taking the steps one or two at a time, in any combination. In how many different ways can you climb a staircase of 10 steps? Explain why the pattern you find appears.

Exercise (Tower of Hanoi)**.** A pile of n discs, each smaller than the one below, must be moved to another peg (three pegs available). One disc moves at a time, and no disc may sit on a smaller one. Let M_n be the minimum number of moves. Explain why $M_{n+1} = 2M_n + 1$, and find a direct formula for M_n.

Exercise. The sum of the first n terms of a sequence is $S_n = 3n^2 + 5n$. Prove that the sequence is arithmetic, and find its first term and common difference.

Textbook Exercises: SPS Course 2.11, Revision Exercise; SPS Course 1.4, Exercise 3 (Extension)